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CITATION:

Goebel, Kazimierz...[et al]. A PROBLEM CONCERNING MAPPINGS WITH CONSTANT DISPLACEMENT (Nonlinear Analysis and Convex Analysis). 数理解析研究所講究録 2002, 1298: 135-140

ISSUE DATE:

2002-12

URL:

<http://hdl.handle.net/2433/42690>

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A PROBLEM CONCERNING MAPPINGS WITH CONSTANT DISPLACEMENT.

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ABSTRACT. We present here an open problem concerning lipschitzian self mappings of closed convex subsets of Banach spaces.

Let X be a Banach space with norm $\|\cdot\|$ and let C be a nonempty, convex closed and bounded subset of X . A lot of attention has been focused recently on the behavior of lipschitzian self mappings of such sets C . Let us recall that the mapping $T : C \rightarrow C$ is *lipschitzian* (satisfies Lipschitz condition) if there exists $k \geq 0$ such that

$$(1) \quad \|Tx - Ty\| \leq k \|x - y\|,$$

for all $x, y \in C$. The smallest k for which (1) holds is said to be the Lipschitz constant for T and is denoted by $k(T)$. If (1) holds we also say that T is k -lipschitzian or that T is of class $\mathcal{L}(k)$, $T \in \mathcal{L}(k)$.

If C is compact then due to the Schauder Fixed Point Theorem any continuous (thus also any lipschitzian) mapping $T : C \rightarrow C$ has a point x satisfying $x = Tx$, a *fixed point* of T . If C is not compact, it is no longer true. The strongest known result due to P. K. Lin and Y. Sternfeld [6] states:

- If C is not compact then for any $k > 1$ there exists a mapping $T : C \rightarrow C$ of class $\mathcal{L}(k)$ such that,

$$(2) \quad d(T) = \inf \{\|x - Tx\| : x \in C\} > 0.$$

The number $d(T)$ defined by (2) is called the minimal displacement of T and mappings T which satisfy (2) are called mappings with positive displacement.

Once we have a lipschitzian mapping T with positive displacement $d = d(T) > 0$ we can define a modified mapping $\tilde{T} : C \rightarrow C$ by

$$\tilde{T}x = x + d \frac{Tx - x}{\|Tx - x\|}.$$

It is easy to observe that \tilde{T} is also lipschitzian but the Lipschitz constant $k(\tilde{T})$ is not necessarily the same as $k(T)$.

This modified mapping has *constant positive displacement* equal d , which means that for all $x \in C$ we have

$$\|x - \tilde{T}x\| = d = d(T) > 0.$$

Date: November 17, 2002.

1991 *Mathematics Subject Classification*. Primary 47H10.

Key words and phrases. lipschitzian mappings, fixed points, mappings with constant displacement, rotative mappings.

Now we can observe that for any $c \in (0, 1]$ the convex combination of the mapping \tilde{T} with the identity mapping I ,

$$\tilde{T}_c = (1 - c)I + c\tilde{T},$$

is also of positive displacement equal cd . Moreover, we have

$$k(\tilde{T}_c) = k((1 - c)I + c\tilde{T}) \leq 1 - c + ck(\tilde{T})$$

and consequently, $\lim_{c \rightarrow 1} k(\tilde{T}_c) = 1$.

The above allows us to formulate an equivalent modification of the Lin Sternfeld result.

- **If C is not compact then for any $k > 1$ there exists a mapping $T : C \rightarrow C$ of class $L(k)$ with constant positive displacement.**

From now on we shall discuss only mappings with constant positive displacement. Suppose $T : C \rightarrow C$ is such a mapping with $d(T) = d > 0$. The iterated mapping $T^2 = T \circ T : C \rightarrow C$ is not necessarily of constant displacement. For any $x \in C$ we have an obvious inequality

$$0 \leq \|T^2x - x\| \leq \|T^2x - Tx\| + \|Tx - x\| = 2d.$$

If $\|T^2x - x\| = 2d$ then the line consisting of two linear segments $[x, Tx]$ and $[Tx, T^2x]$ is isometric to the segment $[x, T^2x]$ and consequently to the interval $[0, 2d]$. If $\|T^2x - x\| < 2d$ it means that the vector $T^2x - Tx$ is in some metric sense "rotated" with respect to the vector $Tx - x$. For this reason it is natural to introduce two coefficients

$$a_-(T) = \inf \left\{ \frac{1}{d} \|T^2x - x\| : x \in C \right\}$$

and

$$a_+(T) = \sup \left\{ \frac{1}{d} \|T^2x - x\| : x \in C \right\}.$$

Intuitively, they represent the *minimal metric rotation* and *global metric rotation*. It is understood in the sense that if $a_-(T) < 2$ then for any $\varepsilon > 0$ there are some $x \in C$ for which the vector $T^2x - Tx$ of length d is *rotated* with respect to the vector $Tx - x$ of the same length in such a way that

$$(3) \quad \|(T^2x - Tx) + (Tx - x)\| < (a_-(T) + \varepsilon)d.$$

If $a_+(T) < 2$, then (3) holds for all $x \in C$. Especially if $a_+(T) = 0$ then T is an *involution*, $T^2 = I$ on C .

There are several open problems and questions concerning mutual relations between constants $k(T)$, $a_-(T)$ and $a_+(T)$. Here is the first observation.

Let $T : C \rightarrow C$ be a mapping of class $\mathcal{L}(k)$ with constant displacement $d(T) = d$. Take any point $x \in C$ and put $u = \frac{1}{2}(Tx + T^2x)$, $v = \frac{1}{2}(x + Tx)$. Then we have

$$\begin{aligned} d = \|u - Tu\| &= \left\| \frac{1}{2}(Tx + T^2x) - Tu \right\| \leq \\ &\leq \frac{1}{2}\|Tx - Tu\| + \frac{1}{2}\|T^2x - Tu\| \leq \\ &\leq \frac{k}{2}\|x - u\| + \frac{k}{2}\|Tx - u\| \leq \\ &\leq \frac{k}{2}\|x - v\| + \frac{k}{2}\|v - u\| + \frac{k}{2}\|Tx - u\| = \\ &= \frac{k}{4}d + \frac{k}{4}\|x - T^2x\| + \frac{k}{4}d = \\ &= \frac{k}{2}d + \frac{k}{4}\|x - T^2x\|. \end{aligned}$$

The conclusion of it can be written in the form

$$\frac{\|x - T^2x\|}{d} \geq 2 \left(\frac{2}{k} - 1 \right)$$

and this shows that the rotation constants and Lipschitz constant of T must satisfy

$$(4) \quad a_+(T) \geq a_-(T) \geq 2 \left(\frac{2}{k(T)} - 1 \right).$$

In other words we have

- If $T : C \rightarrow C$ is a lipschitzian mapping with constant positive displacement then

$$(5) \quad k(T) \geq \frac{4}{a_-(T) + 2}.$$

The above evaluation is probably not sharp. The main open problem connected with mappings of constant displacement can be described as follows.

Problem 1. For any $a \in [0, 2]$ find the value

$$\kappa(a) = \inf \{k : \text{there exists a mapping } T : C \rightarrow C \text{ with } k(T) = k \text{ and } a_-(T) = a\}$$

The evaluation (5) shows that

$$(6) \quad \kappa(a) \geq \frac{4}{a + 2}.$$

The above has been shown without taking into account any geometrical properties of the set C . One can restrict himself to some particular situation of a given set C and define relative function $\kappa_C(a)$. We shall stay here with the general case. However to estimate $\kappa(a)$ from above we have to discuss a concrete construction.

Let $X = C[0, 1]$ with the usual uniform norm and let the set K be defined by

$$K = \{x \in C[0, 1] : 0 = x(0) \leq x(t) \leq x(1) = 1\}.$$

Let e be the identity function on $[0, 1]$, $e(t) \equiv t$. Any function $\alpha \in K$ generates a mapping $T_\alpha : K \rightarrow K$ defined for $x \in K$ by

$$(T_\alpha x)(t) = (\alpha \circ x)(t) = \alpha(x(t)).$$

If α is lipschitzian, so is T_α and we have

$$k(T_\alpha) = k(\alpha) = \sup \left\{ \frac{|\alpha(t) - \alpha(s)|}{|t - s|} : t, s \in [0, 1], t \neq s \right\}.$$

Moreover, since any $x \in K$ takes all the values between 0 and 1, we have

$$\|x - T_\alpha x\| = \max_{t \in [0, 1]} |x(t) - \alpha(x(t))| = \max_{s \in [0, 1]} |s - \alpha(s)| = \|e - \alpha\|.$$

Thus for $\alpha \neq e$, T_α has constant positive displacement $d(T_\alpha) = \|e - \alpha\| > 0$. The iterated mapping $T_\alpha^2 = T_\alpha \circ T_\alpha = T_{\alpha \circ \alpha}$ is of the same type with $k(T_\alpha^2) = k(\alpha \circ \alpha) \leq k(\alpha)^2$ and $d(T_\alpha^2) = \|e - \alpha \circ \alpha\| > 0$. In this case

$$(7) \quad a_+(T_\alpha) = a_-(T_\alpha) = \frac{k(T_\alpha^2)}{k(T_\alpha)} = \frac{\|e - \alpha \circ \alpha\|}{\|e - \alpha\|}.$$

The relation between Lipschitz and rotation constants in this case can be evaluated as follows. There exists at least one point $t \in [0, 1]$ such that

$$|\alpha(\alpha(t)) - \alpha(t)| = \|e - \alpha\| = d(T_\alpha) > 0.$$

Let us assume that at this point $\alpha(\alpha(t)) > \alpha(t)$. The case with converse inequality can be treated the same way. Let t_0 be the minimal point for which the above holds. It means that

$$(8) \quad t_0 = \min \{t : \alpha(\alpha(t)) - \alpha(t) = \|e - \alpha\| = d(T_\alpha)\}.$$

Obviously

$$\alpha(\alpha(t_0)) - \alpha(t_0) = \|e - \alpha\|.$$

Observe that at t_0 we have $\alpha(t_0) \geq t_0$. Indeed, since $\alpha(0) = 0$, if $t_1 = \alpha(t_0) < t_0$ then $\alpha(t_1) = \alpha(\alpha(t_0)) > \alpha(t_0)$ implies existence of a point $t_2 < t_0$ for which $\alpha(t_2) = \alpha(t_0)$. For this point we would have $\alpha(\alpha(t_2)) - \alpha(t_2) = \|e - \alpha\|$ which contradicts (8).

Now, assume that α is lipschitzian with $k(\alpha) = k$. Then, we have

$$\alpha(\alpha(t_0)) - \alpha(t_0) \leq \|\alpha \circ \alpha - \alpha\| = \|T_\alpha \alpha - T_\alpha e\| \leq k \|\alpha - e\| = k(\alpha(t_0) - t_0).$$

Consequently

$$\begin{aligned} \|\alpha \circ \alpha - e\| &\geq \alpha(\alpha(t_0)) - t_0 = [\alpha(\alpha(t_0)) - \alpha(t_0)] + [\alpha(t_0) - t_0] \geq \\ &\geq \left(1 + \frac{1}{k}\right) [\alpha(\alpha(t_0)) - \alpha(t_0)] = \left(1 + \frac{1}{k}\right) \|\alpha - e\| \end{aligned}$$

and finally, in view of (7)

$$(9) \quad 2 \geq a_+(T_\alpha) = a_-(T_\alpha) \geq 1 + \frac{1}{k}.$$

Both inequalities in (9) are sharp. The case $a_+(T_\alpha) = 2$ occurs for any function α of the form

$$\alpha(t) = \begin{cases} (1 + \frac{\varepsilon}{b})t & \text{for } 0 \leq t \leq b < 1 \\ t + \varepsilon & \text{for } b < t \leq 1 - \varepsilon \\ 1 & \text{for } 1 - \varepsilon < t \leq 1 \end{cases},$$

where $b \in (0, 1)$ is arbitrary and ε is sufficiently small.

The equalities $k(\alpha) = k > 1$ and $a_-(T_\alpha) = 1 + \frac{1}{k}$ are satisfied for specially chosen family of functions

$$\alpha_k(t) = \begin{cases} kt & \text{for } 0 \leq t \leq \frac{1}{k} \\ 1 & \text{for } \frac{1}{k} < t \leq 1 \end{cases}.$$

In this setting the family of mappings $T_k = T_{\alpha_k}$, $k \geq 1$ fulfils the conditions

$$k(T_k) = k, \text{ with } d(T_k) = 1 - \frac{1}{k},$$

$$k(T_k^2) = k, \text{ with } d(T_k) = 1 - \frac{1}{k^2},$$

$$(10) \quad a_+(T_\alpha) = a_-(T_\alpha) = 1 + \frac{1}{k}.$$

Comparing (10) with the definition of the function $\kappa(a)$ (see 1) and substituting $a = 1 + \frac{1}{k}$ we get

$$(11) \quad \kappa(a) \leq \frac{1}{a-1}$$

for all $a \in (1, 2]$.

Summing up the estimates (6) and (11) we conclude with

$$(12) \quad \frac{4}{a+2} \leq \kappa(a) \leq \begin{cases} +\infty & \text{if } a \in [0, 1] \\ \frac{1}{a-1} & \text{if } a \in (1, 2] \end{cases}.$$

The gap between the lower and upper bound given by (12) is large. Both inequalities are probably not sharp. The main problem of finding exact formula for $\kappa(a)$ (see Problem 1) leads to some more specific, seemingly simpler, but still remaining without answer partial questions.

Problem 2. Find better then (12) estimate for $\kappa(a)$.

Problem 3. Is $\kappa(a) < \infty$ on $[0, 1]$? If not, then for which $a \in [0, 1]$, $\kappa(a) < \infty$?

In other words. Can one construct an example of a bounded closed convex set C and a lipschitzian mapping with constant positive displacement $T : C \rightarrow C$ such that $a_-(T) \leq 1$? The same with $a_+(T) \leq 1$?

Problem 4. Is $\kappa(0) < \infty$?

More specifically, does there exist a bounded closed and convex set C and a lipschitzian mapping $T : C \rightarrow C$ of constant minimal displacement and such that $a_-(T) = 0$? Replacing $a_-(T)$ in the last question to $a_+(T)$ we obtain the last „exotic” question.

Problem 5. Does there exists a bounded, closed and convex set C for which there is a lipschitzian involution $(T : C \rightarrow C, T^2 = I)$ having constant positive displacement?

The notion of rotation constant and rotative mappings has been introduced by K. Goebel and M. Koter in [2] and [3]. More informations about these notions can be found in an expository article [4] and books [5] and [1].

REFERENCES

- [1] Goebel K., *Concise course on Fixed Point Theorems*, Yokohama Publishers, Yokohama, 2002.
- [2] Goebel K., Koter M., *A remark on nonexpansive mappings*, Canadian Math. Bull. **24**, (1981) 113-115.
- [3] Goebel K., Koter M., *Fixed points of rotative lipschitzian mappings*, Rend. Sem. Mat. Fis. Milano **51** (1981), 145-156.
- [4] Goebel K., Koter-Mórgowska M., *Rotative mappings in metric fixed point theory*, Proc. NACA 98, World Scientific (1999), 150-156.
- [5] Kirk W. A., Sims B., *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht 2001.
- [6] Lin P. K., Sternfeld Y., *Convex sets with the Lipschitz fixed point property are compact*, Proc. Amer. Math. Soc. **93** (1985), 633-639.

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